RESEARCH PAPER

Developing Proficiency with Teaching Algebra in Teacher Working Groups: Understanding the Needs

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This study reports on a professional learning (PL) initiative aimed at establishing a community of practice, through teacher working groups in which teachers can explore and develop their algebraic pedagogical content knowledge (PCK). Here we report on teachers' solutions to three differently represented algebraic problems and explore what the nature of their solutions tells us about their algebraic reasoning and their PCK. The findings showed that most participants favoured only one solution and provided useful insights for the value of teacher working groups in PL activities to develop teachers' algebraic reasoning, understanding, and extend their range of problem-solving strategies.

The present study was developed from a previous professional learning (PL) initiative led by the first author, with a cross-disciplinary team of tertiary academics from mathematics, science, and mathematics and science education. The team established a Peer Learning Circle (PLC), funded by the *University of Tasmania Community of Practice Initiative Program*, focusing on using self-generated external representations in the teaching of mathematics and science. The success of that PLC community of practice suggested a collegial and theoretically grounded means of exploring mathematical concepts needed in teaching and highlighted the potential of PLCs for PL in both schools and higher education (Hatisaru et al., 2020). The current study expands the focus of the mentioned PLC towards developing secondary school teachers' proficiency with algebra teaching (Years 7 to 10) within a community of practice. The algebra focus stems from a research agenda to study the teaching and learning of algebra, in respect of teachers' pedagogical content knowledge (PCK; see, e.g., Ball et al., 2008; Chick & Beswick, 2018).

Algebra learning plays an important role for students in college level studies (e.g., McCallum et al., 2010). Students' algebra learning outcomes are, nevertheless, sometimes poor in both national and international assessments. For instance, in Australia, only 15% of Year 9 Victorian students gave the correct answer to what is regarded as an appropriate-level question: $2 \times (2x - 3) + 2 + ? = 7x - 4$ (Sullivan, 2011). Research studies show that students' algebra learning outcomes can be enhanced through effective forms of instruction that attend to algebraic proficiency, but also suggest that teachers need to be supported in developing such effective instructional practices (e.g., Star et al., 2015).

We aimed to establish a teacher working group in which participant teachers could solve and discuss algebraic problems with an emphasis on student thinking, develop a deeper understanding of algebraic processes and solution strategies, and allow us to examine the effectiveness of teacher working groups as a PL approach. We envisaged regular virtual meetings in which group members would be sent an algebra problem to solve themselves first, and then asked to anticipate how students might solve it, with the solutions used to guide the substance and direction of the following discussions. A workshop held at the *2021 Annual Conference of the Mathematical Association of Tasmania* (*MAT*) (hereafter referred to as the workshop) provided an opportunity to introduce the teacher working group study and start to build a comprehensive understanding of the needs of teachers in algebra. Here, we analyse and

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report the workshop participants' solutions to three algebraic problems and provide some reflections on the possible needs of teachers in teaching algebra.

Types of Algebraic Activity and the Workshop Problems

Effective teaching practices can inspire and develop mathematics learning. Kilpatrick et al. (2001) identify five intertwined strands to achieve what they propose constitutes mathematical proficiency: conceptual understanding (comprehension of mathematical concepts and operations); procedural fluency (skill in carrying out procedures accurately and efficiently); strategic competence (ability to formulate and solve mathematical problems); adaptive reasoning (capacity for logical thought, reflection, and justification); and productive dispositions (seeing mathematics as sensible, useful, and worthwhile). Corresponding to the first four of these elements, the *Australian Curriculum: Mathematics* (*AC: Mathematics*) targets four desirable proficiencies for students as outcomes of studying mathematics: understanding, fluency, problem solving, and reasoning (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2019). The *AC: Mathematics* aims to ensure that students develop these proficiencies within all content domains, including algebra.

Algebra is commonly accepted as an activity (Kieran, 2007) in which two aspects have been distinguished: "(a) algebra as a systematic way of expressing generality and abstraction; and (b) algebra as syntactically guided transformations of symbols" (Kilpatrick et al., 2001, p. 256). According to Kilpatrick et al. these two main aspects of algebra have led to various activities in school algebra, including representational activities, transformational or rule-based activities, and generalizing and justifying activities. This classification by Kilpatrick et al. (2001) is echoed in Kieran's (2007) GTG model, where the activities of school algebra are grouped into three aspects: Generational, Transformational, and Global/meta-level. If teachers possess a better understanding of these algebraic activities, it can benefit both the students and the teachers in creating a better classroom environment to learn. Supporting teachers to meet curricula expectations can sometimes be difficult, however, because of the breadth and complexity of required teacher learning (Kazemi & Franke, 2004). Our workshop aimed to guide teachers to assist their students in formulating, representing, and solving algebraic problems. We provided three problems, according to the algebraic activities of representing, transforming, and generalising and justifying. Each of these is introduced in the relevant sections that follow (Problems #1–3).

Representational Activities of Algebra

The representational activities of algebra involve translating verbal statements into symbolic expressions and equations. Generally, they include "generating (a) equations that represent quantitative problem situations in which one or more of the quantities are unknown, (b) functions describing geometric patterns or numerical sequences, and (c) expressions of the rules governing numerical relationships" (Kilpatrick et al., 2001, pp. 256-257). Facility with representational activities requires both conceptual understanding of mathematical concepts, and ideas stated verbally, and strategic competence to represent statements in algebraic expressions and equations (Kilpatrick et al., 2001).

Problem #1 is an example of a worded problem where there are two unknown quantities: the number of cows (say x) and the number of chickens (say y). The problem may be solved algebraically by generating two equations representing the problem situation: $x + y = 19$ (number of animals) and $4x + 2y = 62$ (number of legs). The two equations can then be solved simultaneously: four times the first equation is $4x + 4y = 76$, and the difference then obtained by subtracting the second equation is $2y = 14$. Therefore, the number of chickens is $y = 7$. As $x = 19 - y$, the number of cows is $x = 12$. There are, of course, other ways of solving the problem, for example using a table, drawing a pictorial model, using a graph (see Tripathi, 2008), or guessing and checking.

> Problem #1: A farmer had 19 animals on his farm - some chickens and some cows. He also knew that there was a total of 62 legs on the animals on the farm. How many of each kind of animal did he have? (Tripathi, 2008)

Transformational Activities of Algebra

Transformational or rule-based activities include collecting like terms, factoring, expanding, substituting, simplifying expressions, and solving equations. In transformational activities, the rules for manipulating algebraic symbols are mainly used to change the form of an expression or equation to an equivalent one (Kilpatrick et al., 2001). In other words, the majority of these types of activities are concerned with changing the symbolic form of an expression or equation in order to maintain equivalence (Kieran, 2007) (e.g., see Figure 1).

Problem #2 (Star & Seifert, 2006)

Find a, if $0.3a + 0.2 = 1.1$

Solution 1:

$$0.3a + 0.2 = 1.1$$
$$0.3a = 1.1 - 0.2$$
$$\frac{0.3a}{0.3} = \frac{0.9}{0.3}$$
$$a = 3$$

Solution 2:

$$0.3a + 0.2 = 0.9 + 0.2$$
$$0.3a = 0.9$$
$$0.3a = 0.3 \times 3$$
$$a = 3$$

Solution 3:

$$0.1(3a + 2) = 0.1 \times 11$$
$$3a + 2 = 11$$
$$3a + 2 = 9 + 2$$
$$3a = 9$$
$$a = 3$$

Solution 4:

$$\frac{0.3a}{0.1} + \frac{0.2}{0.1} = \frac{1.1}{0.1}$$
$$3a + 2 = 11$$
$$3a = 9$$
$$a = 3$$

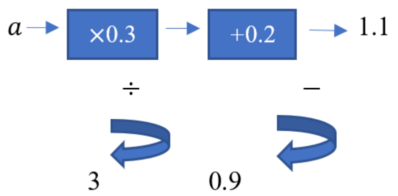
Solution 5a:

$$10 \times (0.3a + 0.2) = 10 \times 1.1$$
$$3a + 2 = 11$$
$$\Delta + 2 = 11$$
$$\Delta = 9$$
$$a = 3$$

Solution 6a:

$$\frac{3}{10}a + \frac{2}{10} = \frac{11}{10}$$
$$\frac{3a + 2}{10} = \frac{11}{10}$$
$$3a + 2 = 11$$
$$3a = 9$$
$$a = 3$$

Solution 7:



$a = 3$

Solution 5b:

$$10 \times (0.3a + 0.2) = 10 \times 1.1$$
$$3a + 2 = 11$$
$$3a = 9$$
$$a = 3$$

Solution 6b:

$$\frac{3}{10}a + \frac{2}{10} = \frac{11}{10}$$
$$3a + 2 = 11$$
$$3a = 9$$
$$a = 3$$

Figure 1. Possible solutions to Problem #2 (reproduced from Hatisaru, 2021).

Charles (2005) suggests the idea of equivalence is one of the *big ideas* in mathematics. Examples of mathematical understanding with algebraic expressions and equations include:

> Algebraic expressions can be named in an infinite number of different but equivalent ways. For example: $2(x - 12) = 2x - 24 = 2x - (28 - 4)$.
>
> A given equation can be represented in an infinite number of different ways that have the same solution. For instance, $3x - 5 = 16$ and $3x = 21$ are equivalent equations; they have the same solution, 7. (Charles, 2005, p. 14).

In order to solve a given equation, the problem solver must engage with equivalent representations of the given expression or equation. Consider the linear equation below:

Problem #2: Find a, if $0.3a + 0.2 = 1.1$ (see Figure 1).

Figure 1 presents a number of different solutions to this equation, where the terms in the equation, and accordingly the equation itself, are represented by its various equivalents. The numerical expression 1.1, for example, is represented as $0.9 + 0.2$ in Solution 2, as 0.1×11 in Solution 3, as $1.1/0.1$ in Solution 4, as 10×1.1 in Solution 5 and as $11/10$ in Solution 6. Similarly, the algebraic expression $0.3a$ is represented with its equivalent forms including $0.1 \times 3a$ (Solution 3) and $3a/10$ (Solution 6). Naming these equivalents yields various solutions, each of which include internal mathematical connections (Hatisaru, 2021).

Facility with transformational activities in algebra is important (McCallum et al., 2010). In these activities, aspects of conceptual understanding and strategic competence interact with each other along with procedural fluency (Kilpatrick et al., 2001). Although there might be a temptation to equate representational activities with the conceptual aspects of algebra and transformational activities with skill-based aspects of algebra, conceptual work and meaning building occur within both types of activities of algebra (Kieran, 2007).

Generalising and Justifying Activities of Algebra

Generalising and justifying activities include problem solving, modelling, justifying, proving, and predicting (Kilpatrick et al., 2001). Although they often use the language and tools of algebra, they are not exclusive to algebra (Kieran, 2007). These activities usually involve examining and interpreting representations that have already been generated or manipulated, and they can generate answers to particular questions or conjectures. In these activities, all aspects of mathematical proficiency come together, but especially adaptive reasoning (Kilpatrick et al., 2001). Problem #3 below illustrates how algebra is used to generalise and justify (see Table 1).

> Problem #3: If you are given the sum and difference of any two numbers, show that you can always find out what the numbers are. (Harper, 1987, as cited in Kieran, 1992)

Table 1
Solution Methods to Problem #3 (adapted from Kieran, 2007)

	Examples
Rhetorical method	Divide the sum by 2, then divide the difference by 2. To get the first number, add the sum divided by 2 to the difference divided by 2. To get the second number, take the difference divided by 2 away from the sum divided by 2.
Diophantine method	Given x is the first number, and y is the second number, assume that $x – y = 2$ and $x + y = 8$. x and y can be found by solving these two equations for x and y, and it is clear this can be applied for any numbers.
Vietan method	Assume the numbers are x and y. m: the sum of x and y. Then, $m = x + y$ n: difference of x and y. Then, $n = x – y$ Add together: $m + n = 2x$ Find x and substitute back for y. That is, $x = (m + n)/2$ and $y = (m – n)/2$

Harper (1987, as cited in Kieran, 2007) used Problem #3 to investigate the stages that algebra students pass through in their development of algebraic symbolism. According to Kieran, Harper interviewed 144 secondary school students and found evidence of three types of solutions identified as being used historically in solving such generalisation questions: the *Rhetorical method*, the *Diophantine method*, and the *Vietan method* (Table 1). It is notable that, while in the *Diophantine method* letters are used to represent unknowns, in the *Vietan method* letters are used for both unknown and given quantities. In the *Rhetorical method*, algebraic symbolism is not used but a procedure that is general is specified (Kieran, 2007).

Teachers' Solutions to the Workshop Problems

As mentioned earlier, the workshop aimed to guide teachers to assist their students in solving algebraic problems representing the three types of algebraic activities presented (GTG, Kieran, 2007). Participants were also asked to what extent they had opportunities to discuss such problems in a PL capacity in their schools, and if so, what they thought about the value of considering such aspects or approaches. At the commencement of the workshop, participants were given a problem sheet and 20–25 minutes to complete it. The sheet included the three problems sketched above and a prompting statement:

> *Find and explain as many different possible solutions to each of the problems as you can. Name the solutions as Solution A, Solution B, Solution C and so on.*

Participants were also given the opportunity to respond the questions below.

> *Assume you are teaching these problems in the class. For each problem identify: (a) the solutions that you would use to solve and why; (b) the solutions that your students might use; and (c) the solutions you hope your students would use.*

Once completed, all participants attached their solutions to the problems (see Table 2 and next page, more details to follow) to a wall in the workshop room and were given time to peruse each other's solutions. This provided participants time to reflect on the variation in solutions of each problem within the group, and upon themselves as teachers of mathematics. We did not formally record the discussions at this point, but there were some rich interactions, which focused predominantly on the solutions which were different to their own. Many participants commented that they had limited time for such in-depth exploration of problems in their usual school-based PL. Nine participant teachers gave their consent for the research, and they were assigned codes P1, P2, P3, etc. to protect their anonymity.

Table 2
Teachers' Solution Strategies to the Workshop Problems

Problem	Solution Strategies	Participants (correct, incomplete, incorrect)
Problem #1	Use a table	P2, P5
	Use simultaneous equations	P1, P4, P5, P7, P8, P9
	Guess and check	P1, P3, P6, P9
	Pictorial	P5
Problem #2	Only one solution	P2, P3, P4, P5, P6, P7, P9
	Two solutions	P1, P8
Problem #3	Using numerical examples	P9
	Diophantine method	P2, P3, P8
	Vietan method	P1, P4, P5, P6, P7

We examined participants' responses according to the solutions presented in the previous section, and also recorded any additional solutions that emerged in the data (P9's solution in Table 3). Table 2 presents the results based on our assessments. We received a total of thirty-three solutions to consider and assess (a few participants provided more than one solution). While most responses included evidence of the types of solutions presented above, a few responses were incomplete, and a few were incorrect. We used traffic light colours to represent them, yellow indicating incomplete, and red indicating incorrect solutions.

In general, the participants solved Problem #1 by using simultaneous equations, and to a lesser extent by the use of a guess and check method (e.g., Figure 2). P1 provided two and P5 provided three different methods to solve the problem, while P4, P6, P7, and P8 gave only one solution. P2's solution was incorrect, and P3 and P9 had incomplete solutions.

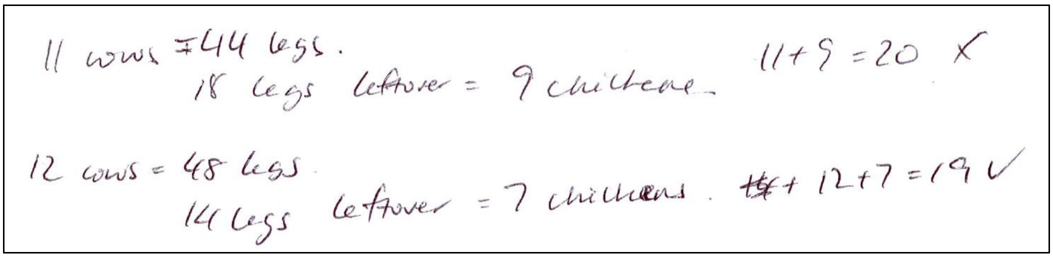
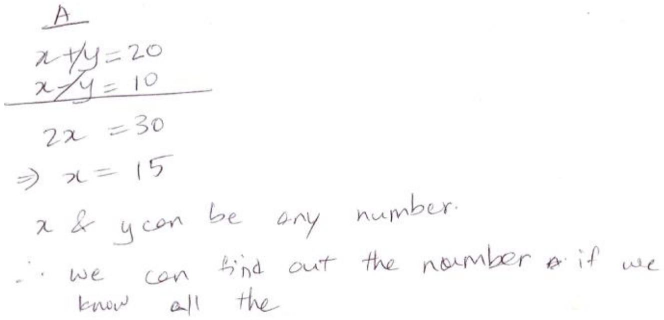
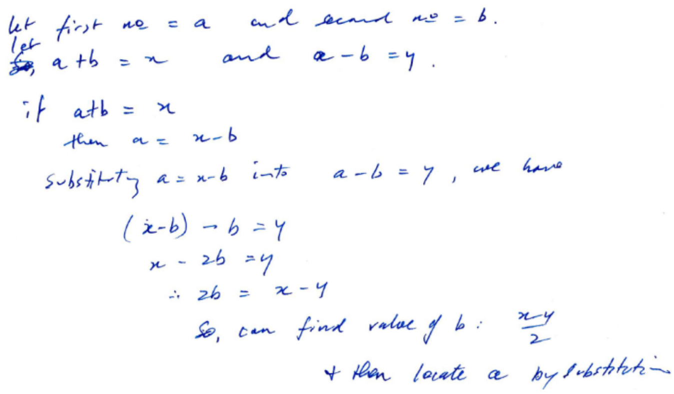
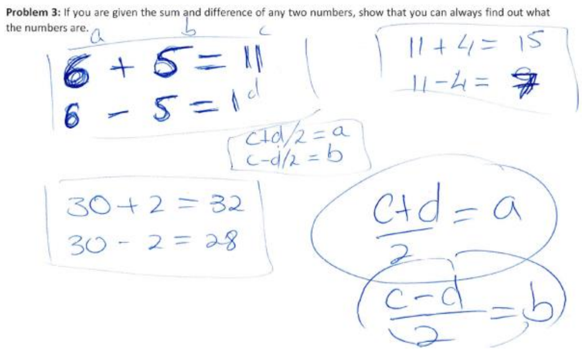


Figure 2. P6's solution to Problem #1.

In responding to Problem #2, seven participants gave only one solution, and their solutions refer to either Solution 1 (five occurrences) or Solution 5 (two occurrences) presented in Figure 1. P1 and P8 solved the equation also by the guess and check method, in addition to using Solution 1. None of the other types of solutions presented in Figure 1 were found in our workshop participants' responses.

Table 3
Example Teacher Solutions to Problem #3

P8's solution: Diophantine method	
P4's solution: Vietan method	

P9's solution: Using numerical examples	

Among the three, Problem #3 seemed to be the most challenging problem to the participants as there were comparatively more incomplete or incorrect solutions. From the responses of the nine participants, we found evidence of the two types of solutions recorded in previous research—the Diophantine and the Vietan methods (see Table 3)—while none of the participants used the Rhetorical method. P9 worked with numerical examples in solving the problem and concluded from these examples (Kieran, 1992) as shown in Table 3. That is, P9 assumed that the numbers are $a = 6$ and $b = 5$. Their sum, c, is 11 then, and their difference, d, is 1. P9 next tried if $a = 11$ and $b = 4$ ($c = 15$, $d = 7$), and next if $a = 30$ and $b = 2$ ($c = 32$, $d = 28$). Based on observations on the sum and difference in each case, P9 concluded the result as: $(c + d)/2 = a$ and $(c – d)/2 = b$. P9 may have had difficulty in using letters to express the general equation (Kieran, 1992), or P9's conceptions of generalisation and justification may be somewhere between "a procedural conception, which derives support from numerical operations, and a structural conception" (p. 407). With the absence of interview data, however, we would be cautious to make these judgements.

Reflections on Teachers' Solutions and Conclusions

It is interesting that for all three problems, the participants' favoured an algebraic response, for example simultaneous equations in Problem #1, and most did not consider additional solutions (maybe time played a part). While such an approach is explicitly prompted by Problem #2, only two participants provided more than one solution, despite the varied possibilities suggested in Figure 1. This may be because they felt that they had exhausted the possibilities for other algebraic manipulation and did not consider the possibility of numerical manipulation. It is, perhaps, not surprising that Problem #3 was the most challenging, because the absence of explicit numerical values makes the problem more abstract. The Vietan method involves pronumerals that are *unknown knowns* (the sum and difference) and *unknown unknowns* (the original two numbers). In the Vietan method case, the method used to find the solution is sufficient to prove that it is the solution, but in other cases, such as in P9's solution, the values have been found but are unproven.

To conclude, this paper analysed the solutions of this sample of teachers to the workshop problems to identify the possible needs of teachers in teaching algebra. Most participants considered only one solution, and this tended to favour more traditional, algebraically procedural methods. Although we have not reported extensively on the discussions that teachers had, they found value in analysing alternative solutions, and identifying the different ways in which algebra is used to represent, transform, and generalise. We anticipate that—with exposure to different forms of algebraic activities, and time to discuss various solutions with their peers—teachers' understanding of algebraic activity, and expectations for students, would grow. We look forward to further developing this PL approach based on these findings.

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